AMD-Vol. 200/MD-Vol. 57, Plastic and Fracture Instabilities in Materials
ASME 1995

STABILITY AND EVOLUTION OF CRACK SYSTEMS IN DISSIPATIVE MATERIALS

A. EL-AZAB and N. M. GHONIEM
Mechanical and Aerospace Engineering Department
University of California, Los Angeles
California 90095-1597

ABSTRACT

Recent trends in manufacturing ceramics in the form of composites have demonstrated the possibility of controlling the energy-of-fracture of the composite by enforcing dissipation mechanisms (e.g., debonding of reinforcements and friction on internal surfaces) over damage zones (extended process zones) surrounding the matrix cracks. The evolution of a damage zone around a matrix crack whether under monotonic loading, cyclic loading or fiber creep conditions proved to have a great impact on matrix crack propagation. A theoretical framework is developed to study the evolution of matrix cracks in ceramic-matrix composites and the associated systems of debonding cracks which are subject to frictional interfacial stresses. The developed theory is an analog extension of the theory of plasticity. It aims at determining the evolution speeds by using the orthogonality relationships for quasi-static evolution of stable crack systems with account for energy dissipation on internal surfaces. The methodology is outlined for ceramic-matrix composites under two different conditions; (i) monotonically increasing external load, and (ii) fiber creep in the bridging zone of the matrix crack under time-independent external load.

INTRODUCTION

In fiber-reinforced ceramic-matrix composites (CMCs), the formation and propagation of a matrix crack is controlled by an *extended* process zone which surrounds it. The size of this zone is dictated by fiber debonding, slip and fiber failure activities in the crack wake. This process zone may extend over the entire matrix crack for small-size matrix flaws which grow during material loading. The concept of crack bridging in ceramic composites evolved around the micro mechanical processes of fiber debonding and slip, and several different types of treatments were devised to deal with the effects of fiber-bridging on the toughening behavior in this class of materials. Reviews of the relevant literature are given by Cox and Marshall (1994) (see also Marshall et al., 1985; Marshall and Cox, 1987; Cox and Marshall, 1991; McCartney, 1992a,b). Under monotonic loading conditions, the evolution of the fiber debonding/slip zone is coupled only with the applied stress as a loading parameter. At elevated temperatures, different kinetic processes (e.g., interface creep in composites with glassy interfaces (Nair and Jakus, 1988; Jakus and Nair, 1988; Nair et al., 1991; Nair and Gwo, 1993) and fiber creep (Begley et al., 1995; El-Azab, 1994) influence the overall response of matrix cracks. Under fiber creep in composite systems with weakly bonded interfaces, evolution of the debonding profile around the matrix crack is expected, even prior to matrix crack propagation under a fixed external load. The continuous evolution of debonding cracks in such material systems results in increased internal surfaces over which energy is dissipated by friction, and thus maintaining the crack tip shielding capability. Previous studies (El-Azab, 1994; Begley et al.,1995) have shown that relaxation of fiber bridging tractions leads to transient response of matrix crack opening and propagation. In particular, the work of El-Azab (1994) showed that, under fiber creep conditions, initially subcritical matrix cracks exhibit a behavior characterized by an incubation period during which the matrix crack opening and the tip stress intensity increase with time. Following this incubation period, a transient and a steady-state growth regime were observed.

Matrix cracking in ceramic-matrix composites and associated fiber debonding and slip represent a damage state which needs to be carefully treated. So far, that fact that matrix and debonding cracks co-exist in the same material, and that these cracks simultaneously evolve in response to the same external load have not been considered in studying the overall cracking behavior of ceramic-matrix composites. The ultimate path to failure, particularly in composite systems with complex fiber architectures, may be determined by the debonding and slip response at the fiber-matrix interface, and a proper treatment of debonding evolution is thus very important in analyzing cracking failure of the composite. During stable growth, it has been assumed that both matrix and debonding cracks are under two different types of loads. Fiber stress acting on an idealized single fiber cell causes interface debonding and/or slip to propagate, while a net stress (applied stress minus bridging stress) acting on the matrix crack faces leads to matrix crack propagation. These two problems were treated separately and coupled only via kinematic constraints (see for example Budiansky et al., 1986; Nair, 1990; Budiansky et al., 1995, where idealized conditions are assumed). Considering the fact that matrix and debond cracks comprise a system of co-existing cracks which simultaneously respond to external loading agents, the growth rates of these cracks with respect to the loading parameters must be simultaneously determined using the same field solution. The main goal of the present work is to develop and outline a formal procedure for determination of the evolution rates of a system of matrix and debonding cracks. These rates are determined starting with the system potential energy functional, which depends on all crack lengths (areas) and loading parameters.

Figure 1 illustrates the problem studied here. A matrix crack which is initially subcritical under the effect a mode I remote loading is considered, along with a system of debonding cracks which develop at the instant of initial matrix crack opening. Unidirectional composites are considered here just to illustrate the method but the method itself is applicable to damage evolution in ceramic composites with any fiber architecture and under different loading conditions. The cases treated are: (i) Time-independent case: evolution of debonding cracks prior to and during matrix crack propagation under monotonic loading, and (ii) Time-dependent case: evolution of debonding cracks prior to and during matrix crack propagation under constant external load and fiber creep in the bridging zone. In case of monotonic loading, debonding cracks evolve depending on a quasi-static loading parameter, which is essentially the applied load. Under fiber creep conditions, the crack system evolves as a function of time and is controled by fiber creep rates. For this purpose, the time-dependent strain energy release rates are derived for the individual cracks, and propagation of any of the cracks is considered to take place at a constant strain energy release rate. In case of no fiber creep, time will be considered in the sense of varying the applied stress. The main features of the approach followed here will be emphasized during the course of formulation.

Following this introduction, a brief outline of the quasi-static evolution problem in multiply-cracked ceramics is formulated. An application to the evolution of crack systems in ceramic-matrix composites under the conditions of monotonic loading and time-independent loading with fiber creep in the bridging zone of a matrix crack will then be given. The manuscript concludes with discussion of different aspects and applicability of the proposed method to treating more complex problems.

QUASI-STATIC EVOLUTION OF CRACK SYSTEMS

Consider a loaded linearly elastic material, which contains a crack population of number N, where N may be considered a function of time to account for the possibility of nucleation of new cracks during loading and/or evolution. This situation is schematically shown in Figure 2. In fiber-reinforced ceramic-matrix composites, debonding cracks appear in the system if the matrix crack is evolving. The assumption of linear elasticity will be relaxed later when creep effects are considered. Assume that crack nucleation and evolution follows the Griffith theory of brittle fracture. Let P denote the potential energy of the of the system under consideration. The potential energy of the cracked domain will generally depend on the number of cracks as well as the individual crack lengths (areas). Let the volume of the domain be denoted by V and its boundary be denoted by ∂V. To illustrate the method, consider an applied traction $T = \lambda T_o$ where λ is a time-dependent loading parameter, and T_o is a fixed spatial traction distribution function. The potential energy of the systems can be written in terms of the strain energy density function, $W(\varepsilon)$, and the applied traction as follows

$$P = \int_V W(\varepsilon_{ij})\,dV - \int_{\partial V} \lambda T_o \tilde{u}\,dS \tag{1}$$

where $W(\varepsilon)$ is a quadratic function of the strain tensor ε_{ij}. $\tilde{u}$ is the displacement vector around the boundary. Let ℓ_i, $i = 1, \ldots, N$, denote the instantaneous lengths (areas) of the individual cracks in the system. The strain energy release rate of the individual cracks can be represented in terms of the rate of change of the system potential energy with respect to the individual crack lengths as follows

$$\Im_i = -\frac{\partial P}{\partial \ell_i}, \tag{2}$$

Let the crack length (area) vector $\bar{\ell}(t)$ represent the evolution for the crack systems, which at $t = 0$ yields the initial conditions. Furthermore, assume that the potential energy of the system and the strain energy release rates for the individual cracks can be computed. The evolution problem can then be stated as follows: find $\bar{\ell}$ such that

$$\Im_i = -\frac{\partial P(\bar{\ell}, \lambda)}{\partial \ell_i}, \quad \dot{\ell}_i = 0 \text{ if } \Im_i < \Im_{ic} \text{ and } \dot{\ell}_i \geq 0 \text{ if } \Im_i = \Im_{ic} \tag{3}$$

where the superimposed dot indicates the time derivative (right-hand derivative) and $\Im_{ic}$ are material constants. For one crack, $\Im \leq \Im_c$ is a convex function in a one-dimensional real space. For a system of N cracks, the vector $\bar{\Im} \leq \bar{\Im}_c$ is also convex in an N-dimension real space, and, by analogy to elastoplasticity, we obtain the normality condition for brittle fracture in the form (Maugin, 1992)

$$\dot{\bar{\ell}} \bullet (\bar{\Im} - \bar{\Im}^*) \geq 0 \tag{4}$$

for every $\bar{\Im} \leq \bar{\Im}_c$ and some arbitrary vector $\bar{\Im}^*$ within the critical hypersurface $\bar{\Im} = \bar{\Im}_c$. The aspects of similarity between elastoplasticity and brittle fracture are explained by Maugin (1992). The following orthogonality relation can easily be drawn from Eq. (4)

$$\dot{\ell}_i \dot{\Im}_i = 0 \tag{5}$$

The result (5) is true with or without summing over i. So, as in plasticity, the problem can be posed in terms of velocities, that is: assuming that the actual state of the system is known, and given the time-rate of change of the loading parameter λ, the system evolution speed vector, $\dot{\bar{\ell}}$, is the unknown of the problem. The problem (5) has the trivial solution $\dot{\bar{\ell}}_i = 0$ if $\bar{\Im} \leq \bar{\Im}_c$ even if $\dot{\bar{\Im}}_i \neq 0$. The non-trivial solution for the problem (5) can be found by considering $\dot{\bar{\Im}}_i = 0$, and hence, by using the Eq. (2), we have

$$\dot{\Im}_i = -\frac{d}{dt}\left[\frac{\partial P(\bar{\ell}, \lambda)}{\partial \ell_i}\right] = 0 \tag{6}$$

Therefore, the evolution speed of the system can be found by rewriting Eq. (6) in the form

$$-\sum_{j=1}^{N} \frac{\partial^2 P(\bar{\ell}, \lambda)}{\partial \ell_i \partial \ell_j} \dot{\ell}_j = \dot{\lambda} \frac{\partial^2 P(\bar{\ell}, \lambda)}{\partial \ell_i \partial \lambda} \quad \text{for } i = 1, \ldots, N \tag{7}$$

The invariance of the strain energy release rate vector during growth, i.e., the condition $\dot{\mathfrak{I}}_i = 0$, corresponds to a fixed yield surface in elastoplasticity (perfect plasticity) with no kinematic or isotropic hardening. However, as far as the overall load-displacement response is concerned, stable crack growth under the influence of dissipative mechanisms might exhibit a hardening character. In brittle fracture, the invariance of the strain energy release rate vector implies invariance of the stress intensity factor vector for the propagating cracks. According to Maugin (1992), the system (7) has at least one solution if the second order partial derivative matrix $\frac{\partial^2 P(\bar{\ell}, \lambda)}{\partial \ell_i \partial \ell_j}$ is positive definite, and this solution is unique if the individual second order derivatives $\frac{\partial^2 P(\bar{\ell}, \lambda)}{\partial \ell_i \partial \ell_j}$ are strictly positive. The result (7) can be easily extended to find the evolution speed for a system with a number M of time-dependent loading parameters ($\bar{\lambda} = \lambda_k$, $k = 1, \ldots, M$). In this case, the potential energy of the system depends on the vector $\bar{\lambda}$, so that $P = P(\bar{\ell}, \bar{\lambda})$, and the system evolution speed can then be found by solving the system

$$-\sum_{j=1}^{N} \frac{\partial^2 P(\bar{\ell}, \bar{\lambda})}{\partial \ell_i \partial \ell_j} \dot{\ell}_j = \sum_{k=1}^{M} \dot{\lambda}_k \frac{\partial^2 P(\bar{\ell}, \bar{\lambda})}{\partial \ell_i \partial \lambda_k} \quad \text{for } i = 1, \ldots, N \tag{8}$$

In general, the evolution relationships (7) and (8) yield finite growth speeds as long as the solvability conditions for these systems are met. The onset of unstable growth can be studied by investigating the properties of the coefficient matrix to the left-hand side of these relationships. Certain physical conditions may affect these coefficients in such a way that matrix ceases to be positive definite, i.e., becomes singular. A discussion on possible extrinsic factors (or bifurcation parameters) which may affect the growth stability of crack systems in the ceramic composite type considered here will be given later. The general results of this section will now be generalized to study the problem of a matrix crack and a system of debonding cracks in ceramic matrix composites in two situations: (i) monotonic loading and (ii) time-independent external loading with fiber creep in the bridging zone.

EVOLUTION OF CRACK SYSTEMS IN CMCS UNDER MONOTONIC LOADING

Consider a ceramic matrix composite with a matrix crack which extends between $-c$ and c along the x-axis, under the influence of a remote loading mode I applied stress σ_a. As shown in Figure 1, a system of debonding cracks develops during the initial opening of the matrix crack. The formation of debonding cracks allows partial slip of fibers close to the matrix crack surface, thus allowing matrix crack opening. Two different models can be used to study the evolution of this system. In the first model the crack system consists of a matrix crack and a "super-debonding" crack. The latter represents all debonding cracks in the systems and has an area A_D. This model will be referred to as the "two-crack model". In the second model, the individual debonding cracks are considered separately, which is referred to as the "multiple-crack model". The two-crack model is convenient to study evolution in composite systems in which fibers in the bridging zone do not fail. This model will be considered first to illustrate the method, and the treatment for the multiple-crack model will be outlined in the end of this section.

Let $\ell_d(x)$ denotes the length of debonding cracks on one side of the matrix crack at point x. In this case, the debonding profile on the entire matrix crack can be represented by

$$\ell_d(x) = \Lambda \bar{\ell}(x) \tag{9}$$

where $\bar{\ell}(x)$ is a fixed spatial profile function (dimensionless) and Λ has units of length. The parameter Λ is a parameter which is a function of the applied load (or time) and whose growth rate is to be determined as a function the external load. The total debonding area around the matrix crack is the integral of the fiber density $f / \pi R^2$ (f = fiber volume fraction and R is the fiber radius) times the local debonding crack area ($2\,(\text{sides}) \times 2\pi R \ell_d(x)$) over the entire matrix crack, which is written as follows

$$A_D = \Lambda \frac{4f}{R} \int_{-c}^{c} \bar{\ell}(x)dx = \Lambda \bar{L} \tag{10}$$

where $\bar{L}$ is a dimensionless geometrical factor. The potential energy of the system is given by

$$P = \int_V W(\varepsilon_{ij})\,dV - (1-f)\,\sigma_a \int_{-c}^{c} \delta_m dx + f \int_{-c}^{c} \sigma_f \delta_f dx + \int_{S_\tau} \tau u \, dS_\tau \tag{11}$$

where δ_m is the matrix crack opening displacement, δ_f is the fiber displacement (unbridged crack opening displacement), u is the slip displacement over the interface debonding crack, σ_f is the fiber stress, τ is the interface friction stress and S_τ is the area over which τ is acting (basically $S_\tau = A_D$). In deriving Eq. (11), the following scenario is assumed. Step 1: assume that both fibers and matrix are cut over the distance $-c$ to c. This allows breaking the problem down to two parts: uncracked domain subject to the remote stress, and a cracked domain subject to σ_a acting on the full cracked area between $-c$ and c. The solution for the uncracked domain does not include any singularities. For the cracked domain (both fiber and matrix are cut) the crack opening displacement is basically given by the solution for unbridged crack problem. Step 2: to reach the final bridged configuration, a distribution σ_f is applied to join fibers from both sides back. Simple energy considerations will then lead to expression (11). During the second step, debonding cracks develop thus dissipating energy by friction (friction term in Eq. (11)). Using the divergence theorem, the strain energy term can be found in terms of the net work done through boundary tractions, and eventually, the variation of the potential energy of the system is written as follows

$$\partial P = -\frac{1}{2}(1-f)\,\sigma_a \int_{-c}^{c} \partial\delta_m dx + \frac{1}{2} f \int_{-c}^{c} \sigma_f \partial\delta_f dx + \frac{1}{2}\int_{S_\tau} \tau\,\partial u\, dS_\tau \tag{12}$$

It can be shown that the second order partial derivatives of the potential energy with respect to matrix crack length and debonding crack area are given by

$$\begin{aligned}
-\frac{\partial^2 P}{\partial c^2} &= \frac{1}{2}(1-f)\,\sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial c^2} dx - \frac{1}{2} f \int_{-c}^{c} \sigma_f \frac{\partial^2 \delta_f}{\partial c^2} dx \\
&\quad - \frac{1}{2}\int_{S_\tau} \tau \frac{\partial^2 u}{\partial c^2} dS_\tau \\
-\frac{\partial^2 P}{\partial A_D^2} &= \frac{1}{2}(1-f)\,\sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial A_D^2} dx - \frac{1}{2}\int_{S_\tau} \tau \frac{\partial^2 u}{\partial A_D^2} dS_\tau \\
-\frac{\partial^2 P}{\partial A_D \partial c} &= \frac{1}{2}(1-f)\,\sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial A_D \partial c} dx - \frac{1}{2}\int_{S_\tau} \tau \frac{\partial^2 u}{\partial A_D \partial c} dS_\tau \\
&= -\frac{\partial^2 P}{\partial c\, \partial A_D}
\end{aligned} \tag{13}$$

where the unbridged crack opening displacement δ_f is assumed to be independent of debonding area, A_D. This assumption will be considered valid when formulating the multiple crack model with no fiber creep. As will be shown later, under fiber creep conditions, only the instantaneous part of the fiber displacement will be considered, which will be dependent on the debonding area. By using Eq. (10) , A_D can be replaced with Λ so that the last two equations in (13) are written as

$$- \frac{\partial^2 P}{\partial \Lambda^2} = \frac{1}{2} (1 - f) \sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial \Lambda^2} dx - \frac{1}{2} \int_{S_\tau} \tau \frac{\partial^2 u}{\partial \Lambda^2} dS_\tau$$

$$- \frac{\partial^2 P}{\partial \Lambda \partial c} = \frac{1}{2} (1 - f) \sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial \Lambda \partial c} dx - \frac{1}{2} \int_{S_\tau} \tau \frac{\partial^2 u}{\partial \Lambda \partial c} dS_\tau \tag{14}$$

$$= - \frac{\partial^2 P}{\partial c \partial \Lambda}$$

For the present case, the result (7) can be rewritten in the form

$$- \frac{\partial^2 P}{\partial c^2} \frac{\partial c}{\partial \sigma_a} - \frac{\partial^2 P}{\partial c \partial \Lambda} \frac{\partial \Lambda}{\partial \sigma_a} = \frac{\partial^2 P}{\partial c \partial \sigma_a}$$

$$- \frac{\partial^2 P}{\partial c \partial \Lambda} \frac{\partial c}{\partial \sigma_a} - \frac{\partial^2 P}{\partial \Lambda^2} \frac{\partial \Lambda}{\partial \sigma_a} = \frac{\partial^2 P}{\partial \Lambda \partial \sigma_a} \tag{15}$$

where the loading parameter λ is replaced with σ_a, and the system evolution speeds $\frac{\partial c}{\partial \sigma_a}$ and $\frac{\partial \Lambda}{\partial \sigma_a}$ are to be determined. The partial derivative $\frac{\partial^2 P}{\partial c \partial \sigma_a}$ can be evaluated as follows

$$\frac{\partial^2 P}{\partial c \partial \sigma_a} = - \frac{1}{2} (1 - f) \int_{-c}^{c} \frac{\partial \delta_m}{\partial c} dx + \frac{1}{2} f \int_{-c}^{c} \left[\frac{\partial \sigma_f}{\partial \sigma_a} \frac{\partial \delta_f}{\partial c} + \sigma_f \frac{\partial^2 \delta_f}{\partial c \partial \sigma_a} \right] dx$$
$$+ \frac{1}{2} \int_{S_\tau} \left[\frac{\partial \tau}{\partial \sigma_a} \frac{\partial u}{\partial c} + \tau \frac{\partial^2 u}{\partial c \partial \sigma_a} \right] dS_\tau \tag{16}$$

The unbridged crack opening profile is given by $\delta_f = \Omega \sigma_a \sqrt{c^2 - x^2}$, where Ω depends on the compliance tensor of the composite. Therefore, $\frac{\partial^2 \delta_f}{\partial c \partial \sigma_a} = \frac{1}{\sigma_a} \frac{\partial \delta_f}{\partial c}$, and the second term to the right-hand side of Eq. (16) can be further simplified. In a similar way, $\frac{\partial^2 P}{\partial \Lambda \partial \sigma_a}$ can be determined as follows

$$\frac{\partial^2 P}{\partial \Lambda \partial \sigma_a} = - \frac{1}{2} (1 - f) \int_{-c}^{c} \frac{\partial \delta_m}{\partial \Lambda} dx + \frac{1}{2} \int_{S_\tau} \left[\frac{\partial \tau}{\partial \sigma_a} \frac{\partial u}{\partial \Lambda} + \tau \frac{\partial^2 u}{\partial \Lambda \partial \sigma_a} \right] dS_\tau \tag{17}$$

The evolution speeds of the system can then by determined by solving Eqs. (15) using the partial derivatives in Eqs. (14), (16) and (17). The system evolution speeds $\frac{\partial c}{\partial \sigma_a}$ and $\frac{\partial \Lambda}{\partial \sigma_a}$ depend on the critical strain energy release rates for the matrix, $\Im_{mc} = 2 (1 - f) \gamma_m$, and debonding $\Im_{dc} = \gamma_d$ (γ_m and γ_d are the fracture energies of matrix and interface, respectively). This can be verified by rewriting two partial derivatives $\frac{\partial^2 P}{\partial c \partial \sigma_a}$ and $\frac{\partial^2 P}{\partial \Lambda \partial \sigma_a}$ in the form

$$\frac{\partial^2 P}{\partial c \partial \sigma_a} = - \frac{\Im_{mc}}{\sigma_a} + \frac{1}{2} f \int_{-c}^{c} \frac{\partial \sigma_f}{\partial \sigma_a} \frac{\partial \delta_f}{\partial c} dx + \frac{1}{2} \int_{S_\tau} \left[\frac{\partial \tau}{\partial \sigma_a} - \frac{\tau}{\sigma_a} \right] \frac{\partial u}{\partial c} dS_\tau + \frac{1}{2} \int_{S_\tau} \tau \frac{\partial^2 u}{\partial c \partial \sigma_a} dS_\tau \tag{18}$$

$$\frac{\partial^2 P}{\partial \Lambda \partial \sigma_a} = - \frac{\Im_{dc} \tilde{L}}{\sigma_a} + \frac{1}{2} \int_{S_\tau} \tau \frac{\partial^2 u}{\partial \Lambda \partial \sigma_a} dS_\tau + \frac{1}{2} \int_{S_\tau} \left[\frac{\partial \tau}{\partial \sigma_a} - \frac{\tau}{\sigma_a} \right] \frac{\partial u}{\partial \Lambda} dS_\tau$$

where use has been made of $\Im_{mc} = - \frac{\partial P}{\partial c}$ and $\Im_{dc} = - \frac{\partial P}{\partial A_D}$. Eqs. (14) through (17) yield the solution to the evolution problem provided that all partial derivatives in Eq. (15) can be determined.

The results can now be specialized to the multiple-crack model. Divide the entire matrix crack (length = $2c$) into N_d intervals. On the interval k, which is bounded by two nodal points x_k and x_{k+1}, consider a super-debonding crack which has an area of

$$A_k = 2\,(\text{sides}) \frac{f}{\pi R^2} \int_{x_k}^{x_{k+1}} 2 \pi R \ell_d (x)\, dx = \Lambda_k \tilde{L}_k \tag{19}$$

where symbols have the same meaning as previously defined. In this case, Eq. (12) can be rewritten for a multiple debonding crack system as follows

$$\partial P = - \frac{1}{2} (1 - f) \sigma_a \int_{-c}^{c} \partial \delta_m dx + \frac{1}{2} f \int_{-c}^{c} \sigma_f \partial \delta_f dx + \frac{1}{2} \sum_{k=1}^{N_d} \int_{S_k} \tau_k \, \partial u_k \, dS_k \tag{20}$$

The second order partial derivatives of the potential energy with respect to matrix crack length and debonding crack areas are then given by

$$- \frac{\partial^2 P}{\partial c^2} = \frac{1}{2} (1 - f) \sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial c^2} dx - \frac{1}{2} f \int_{-c}^{c} \sigma_f \frac{\partial^2 \delta_f}{\partial c^2} dx - \frac{1}{2} \sum_{k=1}^{N_d} \int_{S_k} \tau_k \frac{\partial^2 u_k}{\partial c^2} dS_k$$

$$- \frac{\partial^2 P}{\partial \Lambda_i \partial \Lambda_j} = \frac{1}{2} (1 - f) \sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial \Lambda_i \partial \Lambda_j} dx - \frac{1}{2} \sum_{k=1}^{N_d} \int_{S_k} \tau_k \frac{\partial^2 u_k}{\partial \Lambda_i \partial \Lambda_j} dS_k = - \frac{\partial^2 P}{\partial \Lambda_j \partial \Lambda_i} \quad (\text{including } i = j) \tag{21}$$

$$- \frac{\partial^2 P}{\partial \Lambda_j \partial c} = \frac{1}{2} (1 - f) \sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial \Lambda_j \partial c} dx - \frac{1}{2} \sum_{k=1}^{N_d} \int_{S_k} \tau_k \frac{\partial^2 u_k}{\partial \Lambda_j \partial c} dS_k = - \frac{\partial^2 P}{\partial c \partial \Lambda_j}$$

The evolution speeds can now be found by solving the system of equations

$$-\frac{\partial^2 P}{\partial c^2}\frac{\partial c}{\partial \sigma_a} - \sum_{j=1}^{N_d}\frac{\partial^2 P}{\partial c \partial \Lambda_j}\frac{\partial \Lambda_j}{\partial \sigma_a} = \frac{\partial^2 P}{\partial c \partial \sigma_a}$$

$$-\frac{\partial^2 P}{\partial c \partial \Lambda_i}\frac{\partial c}{\partial \sigma_a} - \sum_{j=1}^{N_d}\frac{\partial^2 P}{\partial \Lambda_i \partial \Lambda_j}\frac{\partial \Lambda_j}{\partial \sigma_a} = \frac{\partial^2 P}{\partial \Lambda_i \partial \sigma_a} \qquad (i = 1, \ldots, N_d) \tag{22}$$

The partial derivatives $\frac{\partial^2 P}{\partial c \partial \sigma_a}$ and $\frac{\partial^2 P}{\partial \Lambda_i \partial \sigma_a}$ $(i = 1, \ldots, N_d)$ can also be evaluated as before.

To determine if the system of cracks is stable at any loading stage, the solvability criterion stated previously must first be checked. If the crack system is stable, then we proceed to determine the speed vector as follows. First is to determine the strain energy release rates for the individual cracks at a particular loading stage. This step determines the number of cracks which are evolving at the given load (see Eq. (3)). In general, if a number N_1 cracks out of a total number N are critical, then an algebraic system of size $N_1 \times N_1$ must be solved to determine N_1 evolution speeds, as follows

$$-\sum_{j=1}^{N_1}\frac{\partial^2 P(\bar{\ell}, \sigma_a)}{\partial \ell_i \partial \ell_j}\frac{\partial \ell_j}{\partial \sigma_a} = \frac{\partial^2 P(\bar{\ell}, \sigma_a)}{\partial \ell_i \partial \sigma_a} \quad \text{for } i = 1, \ldots, N_1 \tag{23}$$

Whenever a crack is sub-critical its velocity is automatically zero. In composite systems of more complex fiber architectures different groups of cracks evolve then saturate over different ranges of loading. This has been confirmed experimentally by Daniel and Anastassopoulos (1995). The multiple-crack model can be useful in studying evolution of debonding cracks in composite systems where fibers are allowed to fail at any stage during loading. In this case, the number N_d can be varied at each load level. Let the system (22) or (23) be rewritten in the form

$$-\mathrm{Q}\,\dot{\bar{\ell}} = \frac{\partial\, \mathrm{q}}{\partial \sigma_a} \tag{24}$$

where the matrix Q and the vector q are given by

$$\mathrm{Q}_{ij} = \frac{\partial^2 P}{\partial \ell_i \partial \ell_j} \quad \text{and} \quad \mathrm{q}_i = \frac{\partial P}{\partial \ell_i} \tag{25}$$

the evolution, $\bar{\ell}$, of the crack system at a load level σ_a^* can then be determined from

$$\bar{\ell} - \bar{\ell}_o = -\int_{\sigma_o}^{\sigma_a^*} \mathrm{Q}^{-1}\frac{\partial\, \mathrm{q}}{\partial \sigma_a}\, d\sigma_a \tag{26}$$

where $\bar{\ell}_o$ is the state of the crack system at load level σ_o.

The formulation so far is exact, and it describes the system consisting of the matrix crack and the debonding cracks in a very general sense using the system potential energy. In order to implement this method, it is required to have a full solution for the unbridged crack opening profile, slip profile, fiber stress profile, and the bridged crack opening profile, which is actually a difficult task. No specific assumptions have been made about the interfacial friction. Thus, the theory presented so far is amenable to different simplifying assumptions as far as the interfacial friction is concerned. A Coulomb-type friction or a constant friction stress model can be used for further development of the solution.

EVOLUTION UNDER FIBER CREEP CONDITIONS

In the previous section, it is assumed that no time-dependent effects are involved and the system quasi-static evolution rates are determined with respect to a monotonically increasing applied load. In this section, the same problem is solved under constant applied stress with fiber creep in the bridging zone of the matrix crack. The formulation here starts with global energy balance relationships for a system of evolving cracks. In addition to the frictional dissipation, viscoelastic effects associated with fiber creep result in viscous dissipation. The material temperature will be assumed independent of time, so that thermal energy terms can be excluded.

Consider a viscoelastic body with an evolving crack population of instantaneous areas $A_k(t)$ ($k=1,..N$). The energy balance for this crack system can be written in the form

$$\sum_{k=1}^{N} \dot{A}_k \Im_{kc} + \int_V \sigma_{ij}(t)\,\dot{\varepsilon}_{ij}(t)\,dV = \int_{\partial V} T_i \dot{\bar{u}}_i(t)\,dS \tag{27}$$

in which $\Im_{kc}$ are the critical strain energy release rates of the individual cracks, the volume integral represents the rate of change of stored mechanical energy of the system (viscous dissipation is included), and the surface integral represents the rate of work done by boundary tractions. The boundary traction is assumed to be time-independent. However, the formulation can easily be extended to account for time-dependent traction boundary conditions. The stress, strain, displacement fields are generally time-dependent. In the present case, the boundary of the material is considered to be the sum of all crack areas, part of which is undergoing pure mode II opening under the influence of a friction stress which may be just a constant or dependent on the spatial position over the surface. In the absence of crack growth, the relationship (27) is an identity. For an elastic material, the volume integral is essentially the rate of change of the elastic strain energy. By breaking strain and the displacement fields into instantaneous (elastic) and remainder parts, it has been shown by Golden and Graham (1990) that, in Eq. (27), the terms associated with non-instantaneous strain and displacement parts cancel from both sides of the balance relationship, which reduces Eq. (27) to

$$\sum_{k=1}^{N} \dot{A}_k \Im_{kc} + \int_V \sigma_{ij}(t)\,\dot{\varepsilon}^e_{ij}(t)\,dV = \int_{\partial V} T_i \dot{\bar{u}}^e_i(t)\,dS \tag{28}$$

where the superscript e denotes the elastic (instantaneous) parts of the strain and displacement fields. The relationship (28) has the form of the energy balance during quasi-static crack propagation in an elastic material, with the volume integral representing the rate of change of elastic strain energy. Using the divergence theorem, that particular term can be rewritten in the form

$$\int_V \sigma_{ij}(t)\,\dot{\varepsilon}^e_{ij}(t)\,dV = \frac{1}{2}\frac{d}{dt}\int_V \sigma_{ij}(t)\,\varepsilon^e_{ij}(t)\,dV = \frac{1}{2}\frac{d}{dt}\int_{\partial V} T_i \bar{u}^e_i(t)\,dS \tag{29}$$

Therefore, Eq. (28) can be reduced to

$$\sum_{k=1}^{N} \dot{A}_k \Im_{kc} = \frac{1}{2}\int_{\partial V} T_i \dot{\bar{u}}^e_i(t)\,dS \tag{30}$$

which can be rewritten in the form

$$\sum_{k=1}^{N} \partial A_k \Im_{kc} = \frac{1}{2}\int_{\partial V} T_i \partial\bar{u}^e_i(t)\,dS \tag{31}$$

By differentiating both sides of Eq. (31) with respect to A_j, we get

$$\Im_{jc} = \frac{1}{2}\int_{\partial V} T_i \frac{\partial \bar{u}^e_i(t)}{\partial A_j}\,dS \tag{32}$$

which means that a functional $\Re$ can be defined such that the strain energy release rate for a particular crack is defined by

$$\Im_j = -\frac{\partial \Re}{\partial A_j} \tag{33}$$

where the functional $\Re$ has the form

$$\Re = \frac{1}{2} \int_V \sigma_{ij} \varepsilon^e_{ij} \, dV - \int_{\partial V} T_i \bar{u}^e_i \, dS \tag{34}$$

The results (33) and (34) have the same forms for time-dependent boundary tractions. These two results are analogous to Eqs. (2) and (1), respectively. Therefore, the evolution problem described by Eq. (3) can now be applicable with P replaced by $\Re$. In addition, the results (5) through (8) have the same form under fiber creep conditions, hence the system evolution can be studied in terms of the partial derivatives of the functional $\Re$.

The analysis presented in this section aims at determining the effects of fiber creep in the bridging zone on the stability and propagation of the matrix crack, which motivates considering a time-independent external load. Consider a unidirectional composite which has a matrix crack extending between $-c$ and c along the x-axis, under the influence of a time-independent remote loading mode I applied stress σ_a. Using the arguments presented in the previous section, the variation of the functional $\Re$ can be written in the form

$$\partial\Re = -\frac{1}{2}(1 - f)\sigma_a \int_{-c}^{c} \partial\delta_m \, dx + \frac{1}{2} f \int_{-c}^{c} \sigma_f \partial\delta^e_f \, dx + \frac{1}{2} \int_{S_\tau} \tau \, \partial u^e \, dS_\tau \tag{35}$$

where all terms have been previously defined (see Eq. (11)). In the previous section, although σ_f and τ are treated as external tractions only σ_a is considered to be the only independent loading parameter. However, σ_f and τ are implicitly treated as external loads in the sense of having non-zero partial derivatives with respect to σ_a (see Eqs. (16) and (17)). Here, σ_f and τ are explicitly treated as external loading parameters, and Eq. (8), rather than (7) is used to study the system evolution. The reason this is pursued here is that fiber creep leads to variations in the fiber stress and (generally) in the interfacial friction stress eventhough the external stress is kept fixed. It is to be noted here that in general σ_f, δ^e_f and τ, respectively, should be written as $\sigma_f = \sigma_f(x, t; \sigma_a, c)$, $\delta^e_f = \delta^e_f(x, t; \sigma_a, c)$ and $\tau = \tau(x, t, z; \sigma_a, c)$, where z is a position coordinate along the axial direction of each fibers over its debonded zone. If τ is chosen to be a material property, it will not be considered a loading parameter, and the results will be simpler as shown in the end of this section.

To carry the analysis further, a multiple-crack model will be considered with multiple loading parameters, in which the integral terms containing σ_f and τ in Eq. (35) are discretized. The reason is that the profiles of the relevant stresses and displacements can not be described by the same shape functions at all times. To justify this step, Figures 3 and 4 show representative evolution results for fiber stress and debonding profiles, respectively, which are calculated using the present method under certain simplifying assumptions, which will be discussed later. However, based on these two figure, it is clear that the shape of the fiber stress and debonding profiles change dramatically during matrix crack propagation.

Let the matrix crack (length = $2c$) be divided into N_d intervals. On the interval k, consider a super-debonding crack which has an area of

$$A_k(t) = 2(\text{sides}) \frac{f}{\pi R^2} \int_{x_k}^{x_{k+1}} 2\pi R \ell_d(x, t) \, dx = \Lambda_k(t) \bar{L}_k \tag{36}$$

where, in the present case, Λ_k is a function of time. Rewrite Eq. (35) in the form

$$\begin{aligned} \partial\Re = & -\frac{1}{2}(1 - f)\sigma_a \int_{-c}^{c} \partial\delta_m \, dx \\ & + \frac{1}{2} f \sum_{k=1}^{N_d} \int_{x_k}^{x_{k+1}} \sigma_k \partial\delta_k \, dx + \frac{1}{2} \sum_{k=1}^{N_d} \int_{S_\tau} \tau_k \, \partial u_k \, dS_\tau \end{aligned} \tag{37}$$

where σ_k is the fiber stress over the interval k, and δ_k and u_k are, respectively, the elastic fiber and slip displacements over the same interval. For the sake of simplicity, the number of intervals for the second and third terms in Eq. (37) is taken to be the same. Now, the functional $\Re$ takes the form $\Re = \Re(c, A_k, \sigma_k(t), \tau_k(t), \ldots)$. The following partial derivatives can be immediately obtained

$$
\begin{aligned}
-\frac{\partial^2 \Re}{\partial c^2} &= \frac{1}{2}(1-f)\sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial c^2}\, dx - \frac{1}{2} f \sum_{k=1}^{N_d} \int_{x_k}^{x_{k+1}} \sigma_k \frac{\partial^2 \delta_k}{\partial c^2}\, dx \\
&\quad - \frac{1}{2} \sum_{k=1}^{N_d} \int_{S_k} \tau_k \frac{\partial^2 u_k}{\partial c^2}\, dS_k \\
-\frac{\partial^2 \Re}{\partial \Lambda_i \partial \Lambda_j} &= \frac{1}{2}(1-f)\sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial \Lambda_i \partial \Lambda_j}\, dx - \frac{1}{2} f \sum_{k=1}^{N_d} \int_{x_k}^{x_{k+1}} \sigma_k \frac{\partial^2 \delta_k}{\partial \Lambda_i \partial \Lambda_j}\, dx \\
&\quad - \frac{1}{2} \sum_{k=1}^{N_d} \int_{S_k} \tau_k \frac{\partial^2 u_k}{\partial \Lambda_i \partial \Lambda_j}\, dS_k = -\frac{\partial^2 \Re}{\partial \Lambda_j \partial \Lambda_i} \quad (\text{including } i = j) \\
-\frac{\partial^2 \Re}{\partial \Lambda_j \partial c} &= \frac{1}{2}(1-f)\sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial \Lambda_j \partial c}\, dx - \frac{1}{2} f \sum_{k=1}^{N_d} \int_{x_k}^{x_{k+1}} \sigma_k \frac{\partial^2 \delta_k}{\partial \Lambda_j \partial c}\, dx \\
&\quad - \frac{1}{2} \sum_{k=1}^{N_d} \int_{S_k} \tau_k \frac{\partial^2 u_k}{\partial \Lambda_j \partial c}\, dS_k = -\frac{\partial^2 \Re}{\partial c \partial \Lambda_j}
\end{aligned}
\tag{38}
$$

Let the fiber stress over the interval k be represented in the form $\sigma_k = \Sigma_k^{\sigma}(t)\,\tilde{\sigma}_k$, where $\Sigma_k^{\sigma}(t)$ is a time-dependent amplitude and $\tilde{\sigma}_k$ is a shape function. Similarly, let τ_k be represented by the product of a time-dependent amplitude $\Sigma_k^{\tau}(t)$ and a shape function $\tilde{\tau}_k$. In this case, Eq. (37) can be rewritten in the form

$$
\begin{aligned}
\partial \Re &= -\frac{1}{2}(1-f)\sigma_a \int_{-c}^{c} \partial \delta_m\, dx \\
&\quad + \frac{1}{2} f \sum_{k=1}^{N_d} \Sigma_k^{\sigma} \int_{x_k}^{x_{k+1}} \tilde{\sigma}_k \partial \delta_k\, dx + \frac{1}{2} \sum_{k=1}^{N_d} \Sigma_k^{\tau} \int_{S_\tau} \tilde{\tau}_k\, \partial u_k\, dS_\tau
\end{aligned}
\tag{39}
$$

Hence, the amplitudes $\Sigma_k^{\sigma}(t)$ and $\Sigma_k^{\tau}(t)$ are the time-dependent loading parameters. Having considered this, the functional $\Re$ now has the form $\Re = \Re(c, A_k, \bar{\Sigma}^{\sigma}(t), \bar{\Sigma}^{\tau}(t))$, and the system evolution can be studied by modifying Eq. (8) to include these vectors, which yields

$$
-\sum_{j=1}^{N} \frac{\partial^2 \Re(\bar{\ell}, \bar{\Sigma}^{\sigma}, \bar{\Sigma}^{\tau})}{\partial \ell_i \partial \ell_j}\, \dot{\ell}_j = \sum_{k=1}^{N_d} \left[\dot{\Sigma}_k^{\sigma} \frac{\partial^2 \Re(\bar{\ell}, \bar{\Sigma}^{\sigma}, \bar{\Sigma}^{\tau})}{\partial \ell_i \partial \Sigma_k^{\sigma}} + \dot{\Sigma}_k^{\tau} \frac{\partial^2 \Re(\bar{\ell}, \bar{\Sigma}^{\sigma}, \bar{\Sigma}^{\tau})}{\partial \ell_i \partial \Sigma_k^{\tau}} \right] \quad \text{for } i = 1, \ldots, N \tag{40}
$$

where N is the total number of cracks in the system which consists of the matrix crack plus a number N_d of debonding cracks, i.e., $N = N_d + 1$, and the vector $\bar{\ell} \equiv (c, A_j;\ j = 1, \ldots, N_d)$. In order to determine the evolution speed vector, $\dot{\bar{\ell}}$, it is further required to find expressions for the partial derivatives $\frac{\partial^2 \Re(\bar{\ell}, \bar{\Sigma}^{\sigma}, \bar{\Sigma}^{\tau})}{\partial \ell_i \partial \Sigma_k^{\sigma}}$ and

$\frac{\partial^2 \Re(\bar{\ell}, \bar{\Sigma}^\sigma, \bar{\Sigma}^\tau)}{\partial \ell_i \partial \Sigma_k^\tau}$. The rate vectors $\dot{\Sigma}_k^\sigma$ and $\dot{\Sigma}_k^\tau$ are also assumed to be available, as will be explained later. Using expression (39), the two partial derivatives can be written as follows

$$\frac{\partial^2 \Re}{\partial c \partial \Sigma_j^\sigma} = -\frac{1}{2}(1-f)\sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial c \partial \Sigma_j^\sigma} dx + \frac{1}{2} f \int_{x_j}^{x_{j+1}} \bar{\sigma}_j \frac{\partial \delta_j}{\partial c} dx$$

$$+ \frac{1}{2} f \sum_{k=1}^{N_d} \Sigma_k^\sigma \int_{x_k}^{x_{k+1}} \bar{\sigma}_k \frac{\partial^2 \delta_k}{\partial c \partial \Sigma_j^\sigma} dx + \frac{1}{2} \sum_{k=1}^{N_d} \Sigma_k^\tau \int_{S_k} \bar{\tau}_k \frac{\partial^2 u_k}{\partial c \partial \Sigma_j^\sigma} dS_k$$

$$\frac{\partial^2 \Re}{\partial c \partial \Sigma_j^\tau} = -\frac{1}{2}(1-f)\sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial c \partial \Sigma_j^\tau} dx + \frac{1}{2} f \sum_{k=1}^{N_d} \Sigma_k^\sigma \int_{x_k}^{x_{k+1}} \bar{\sigma}_k \frac{\partial^2 \delta_k}{\partial c \partial \Sigma_j^\tau} dx$$

$$+ \frac{1}{2} \int_{S_j} \bar{\tau}_j \frac{\partial u_j}{\partial c} dS_j + \frac{1}{2} \sum_{k=1}^{N_d} \Sigma_k^\tau \int_{S_k} \bar{\tau}_k \frac{\partial^2 u_k}{\partial c \partial \Sigma_j^\tau} dS_k$$

$$\frac{\partial^2 \Re}{\partial \Lambda_i \partial \Sigma_j^\sigma} = -\frac{1}{2}(1-f)\sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial \Lambda_i \partial \Sigma_j^\sigma} dx + \frac{1}{2} f \int_{x_j}^{x_{j+1}} \bar{\sigma}_j \frac{\partial \delta_j}{\partial \Lambda_i} dx$$

$$+ \frac{1}{2} f \sum_{k=1}^{N_d} \Sigma_k^\sigma \int_{x_k}^{x_{k+1}} \bar{\sigma}_k \frac{\partial^2 \delta_k}{\partial \Lambda_i \partial \Sigma_j^\sigma} dx + \frac{1}{2} \sum_{k=1}^{N_d} \Sigma_k^\tau \int_{S_k} \bar{\tau}_k \frac{\partial^2 u_k}{\partial \Lambda_i \partial \Sigma_j^\sigma} dS_k$$

$$\frac{\partial^2 \Re}{\partial \Lambda_i \partial \Sigma_j^\tau} = -\frac{1}{2}(1-f)\sigma_a \int_{-c}^{c} \frac{\partial^2 \delta_m}{\partial \Lambda_i \partial \Sigma_j^\tau} dx + \frac{1}{2} f \sum_{k=1}^{N_d} \Sigma_k^\sigma \int_{x_k}^{x_{k+1}} \bar{\sigma}_k \frac{\partial^2 \delta_k}{\partial \Lambda_i \partial \Sigma_j^\tau} dx$$

$$+ \frac{1}{2} \int_{S_j} \bar{\tau}_j \frac{\partial u_j}{\partial \Lambda_i} dS_j + \frac{1}{2} \sum_{k=1}^{N_d} \Sigma_k^\tau \int_{S_k} \bar{\tau}_k \frac{\partial^2 u_k}{\partial \Lambda_i \partial \Sigma_j^\tau} dS_k \tag{41}$$

Define the following matrices

$$Q_{ij} = \frac{\partial^2 \Re}{\partial \ell_i \partial \ell_j}, \quad Q_{ik}^\sigma = \frac{\partial^2 \Re}{\partial \ell_i \partial \Sigma_k^\sigma}, \quad \text{and} \quad Q_{ik}^\tau = \frac{\partial^2 \Re}{\partial \ell_i \partial \Sigma_k^\tau} \tag{42}$$

$$\text{where} \quad \Re = \Re(\bar{\ell}, \bar{\Sigma}^\sigma, \bar{\Sigma}^\tau)$$

Eq. (40) can then be cast in the form

$$-Q\,\dot{\bar{\ell}} = Q^\sigma \dot{\bar{\Sigma}}^\sigma + Q^\tau \dot{\bar{\Sigma}}^\tau \tag{43}$$

and the evolution, $\bar{\ell}(t)$, of the crack system at time t is given by

$$\bar{\ell}(t) - \bar{\ell}(t_o) = -\int_{t_o}^{t} Q^{-1}(t')\left[Q^\sigma(t')\,\dot{\bar{\Sigma}}^\sigma(t') + Q^\tau(t')\,\dot{\bar{\Sigma}}^\tau(t')\right] dt' \tag{44}$$

where $\bar{\ell}(t_o)$ is the system state at time t_o (assumed known).

The friction stress τ is strongly dependent on the micro mechanisms of interface debonding and the mode of opening of debond cracks. It is also dependent on the residual stresses in the composite and whether or not the composite includes an interphase between the fiber and the matrix. As far as modeling the interfacial friction is concerned, two trends appeared in the literature; a constant friction model which assumes that interfacial friction is dominated by surface roughness and is purely a material property, and a Coulomb friction model in which the interfacial friction stress depends on the normal pressure at the fiber-matrix interface. Under the assumption that τ is a material property, it can be taken out of the friction integrals in Eqs. (37), (38), (39) and (41). Also, the vector $\dot{\bar{\Sigma}}^\tau$ becomes identically zero. In this case, the results can be summarized as follows. Eq. (38) reduces to

$$-\frac{\partial^2\Re}{\partial c^2} = \frac{1}{2}(1-f)\sigma_a\int_{-c}^{c}\frac{\partial^2\delta_m}{\partial c^2}\,dx - \frac{1}{2}f\sum_{k=1}^{N_d}\int_{x_k}^{x_{k+1}}\sigma_k\frac{\partial^2\delta_k}{\partial c^2}\,dx - \frac{\tau}{2}\sum_{k=1}^{N_d}\int_{S_k}\frac{\partial^2 u_k}{\partial c^2}\,dS_k$$

$$-\frac{\partial^2\Re}{\partial\Lambda_i\partial\Lambda_j} = \frac{1}{2}(1-f)\sigma_a\int_{-c}^{c}\frac{\partial^2\delta_m}{\partial\Lambda_i\partial\Lambda_j}\,dx - \frac{1}{2}f\sum_{k=1}^{N_d}\int_{x_k}^{x_{k+1}}\sigma_k\frac{\partial^2\delta_k}{\partial\Lambda_i\partial\Lambda_j}\,dx - \frac{\tau}{2}\sum_{k=1}^{N_d}\int_{S_k}\frac{\partial^2 u_k}{\partial\Lambda_i\partial\Lambda_j}\,dS_k = -\frac{\partial^2\Re}{\partial\Lambda_j\partial\Lambda_i}\quad(\text{including } i = j)$$

$$-\frac{\partial^2\Re}{\partial\Lambda_j\partial c} = \frac{1}{2}(1-f)\sigma_a\int_{-c}^{c}\frac{\partial^2\delta_m}{\partial\Lambda_j\partial c}\,dx - \frac{1}{2}f\sum_{k=1}^{N_d}\int_{x_k}^{x_{k+1}}\sigma_k\frac{\partial^2\delta_k}{\partial\Lambda_j\partial c}\,dx - \frac{\tau}{2}\sum_{k=1}^{N_d}\int_{S_k}\frac{\partial^2 u_k}{\partial\Lambda_j\partial c}\,dS_k = -\frac{\partial^2\Re}{\partial c\partial\Lambda_j} \tag{45}$$

Eq. (39) reduces to

$$\partial\Re = -\frac{1}{2}(1-f)\sigma_a\int_{-c}^{c}\partial\delta_m\,dx + \frac{1}{2}f\sum_{k=1}^{N_d}\Sigma_k^{\sigma}\int_{x_k}^{x_{k+1}}\bar{\sigma}_k\partial\delta_k\,dx + \frac{\tau}{2}\sum_{k=1}^{N_d}\int_{S_\tau}\partial u_k\,dS_\tau \tag{46}$$

Eq. (40) reduces to

$$-\sum_{j=1}^{N}\frac{\partial^2\Re(\bar{\ell},\bar{\Sigma}^{\sigma})}{\partial\ell_i\partial\ell_j}\,\dot{\ell}_j = \sum_{k=1}^{N_d}\dot{\Sigma}_k^{\sigma}\,\frac{\partial^2\Re(\bar{\ell},\bar{\Sigma}^{\sigma})}{\partial\ell_i\partial\,\Sigma_k^{\sigma}} \tag{47}$$

and Eq. (41) reduces to

$$\frac{\partial^2\Re}{\partial c\partial\,\Sigma_j^{\sigma}} = -\frac{1}{2}(1-f)\sigma_a\int_{-c}^{c}\frac{\partial^2\delta_m}{\partial c\partial\,\Sigma_j^{\sigma}}\,dx + \frac{1}{2}f\int_{x_j}^{x_{j+1}}\bar{\sigma}_j\frac{\partial\delta_j}{\partial c}\,dx + \frac{1}{2}f\sum_{k=1}^{N_d}\Sigma_k^{\sigma}\int_{x_k}^{x_{k+1}}\bar{\sigma}_k\frac{\partial^2\delta_k}{\partial c\partial\,\Sigma_j^{\sigma}}\,dx + \frac{\tau}{2}\sum_{k=1}^{N_d}\int_{S_k}\frac{\partial^2 u_k}{\partial c\partial\,\Sigma_j^{\sigma}}\,dS_k$$

$$\frac{\partial^2\Re}{\partial\Lambda_i\partial\,\Sigma_j^{\sigma}} = -\frac{1}{2}(1-f)\sigma_a\int_{-c}^{c}\frac{\partial^2\delta_m}{\partial\Lambda_i\partial\,\Sigma_j^{\sigma}}\,dx + \frac{1}{2}f\int_{x_j}^{x_{j+1}}\bar{\sigma}_j\frac{\partial\delta_j}{\partial\Lambda_i}\,dx + \frac{1}{2}f\sum_{k=1}^{N_d}\Sigma_k^{\sigma}\int_{x_k}^{x_{k+1}}\bar{\sigma}_k\frac{\partial^2\delta_k}{\partial\Lambda_i\partial\,\Sigma_j^{\sigma}}\,dx + \frac{\tau}{2}\sum_{k=1}^{N_d}\int_{S_k}\frac{\partial^2 u_k}{\partial\Lambda_i\partial\,\Sigma_j^{\sigma}}\,dS_k \tag{48}$$

The matrix Q_{ik}^{τ} reduces to zero, and Eq. (43) reduces to

$$-\,Q\,\dot{\bar{\ell}} = Q^{\sigma}\,\dot{\bar{\Sigma}}^{\sigma} \tag{49}$$

and the evolution vector, $\bar{\ell}(t)$, is given by

$$\bar{\ell}(t) - \bar{\ell}(t_o) = -\int_{t_o}^{t}Q^{-1}(t')\,Q^{\sigma}(t')\,\dot{\bar{\Sigma}}^{\sigma}(t')\,dt' \tag{50}$$

SOLUTION PROCEDURE

The method outlined in the previous two sections assume that the functionals P and $\Re$ and their second order partial derivatives with respect to the instantaneous loading parameters and crack areas can be determined (see, for example, Eqs. (45) and (48)). This implies that the matrix crack opening profile, $\delta_m(x;\bar{\ell},\bar{\lambda})$, the fiber displacement, $\delta_f(x;\bar{\ell},\bar{\lambda})$, and its instantaneous part, the fiber stress profile, $\sigma_f(x,t)$ and its time-derivative (and $\dot{\Sigma}_k^{\sigma}$), the slip displacement profile, $u(\bar{r};\bar{\ell},\bar{\lambda})$, and its instantaneous part and the interfacial friction stress profile, $\tau(\bar{r};\bar{\ell},\bar{\lambda})$, (and $\dot{\Sigma}_k^{\tau}$) are available at any system state ($\bar{r}$ is the position vector describing the debond surface, $\bar{\ell}$ is the crack system evolution, and $\bar{\lambda}$ is the loading parameters vector). Moreover, the relationships among these field variables are such that the involved derivatives are obtainable. This is actually the difficult part as far as the implementation of the current method is concerned. In the worst case, a numerical treatment may be needed for implementation, which is yet to be investigated. A growth rate equation for a matrix crack under a time-varying bridging traction which was derived by El-Azab (1994) is quoted here to confirm the fact that a complete field solution is required to obtain an evolution speed. This growth equation has the form

$$\frac{a^{-1.5}\sqrt{\pi}}{8}K_{IC} = \int_{-a}^{a}\frac{\frac{\partial\sigma_B(x,t;a)}{\partial a}}{\sqrt{a^2-x^2}}dx + \frac{1}{\dot{a}}\int_{-a}^{a}\frac{\frac{\partial\sigma_B(x,t;a)}{\partial t}}{\sqrt{a^2-x^2}}dx \quad (51)$$

where the matrix crack extended between $-a$ and a on the x-axis, $\dot{a}$ is the crack speed, and K_{IC} is the critical stress intensity factor. The growth rate $\dot{a}$ can be found using such a relationship if the bridging stress profile, $\sigma_B(x,t;c)$, is available.

The steps which must be taken to determine the growth stability and growth rates are as follows. First, the strain energy release rates for the individual cracks must be found in order to determined which cracks are evolving. Second, the coefficient matrix must be checked to determine the growth stability, which may be affected by extrinsic factors such as finite fiber strength, fiber rupture due to excessive creep, residual thermal stresses in the composite, notch size if the matrix crack, or damage in general, is evolving from that notch, specimen size relative to the dominant matrix crack size, etc. These parameters may act in such a way that growth stability exhibit bifurcation at different applied load ranges, or at different evolution stages. So, generally speaking, in addition to their dependence of crack length and loading parameter vectors, the functionals P and $\Re$ depend on a bifurcation parameter vector $\bar{\omega}$ as well, i.e., $P = P(\bar{\ell},\bar{\lambda};\bar{\omega})$ and $\Re = \Re(\bar{\ell},\bar{\lambda};\bar{\omega})$. In case of fiber creep, the constant applied load may itself be a bifurcation parameter as well. Third, the growth rates can be determined, depending on the circumstances, using Eq. (15), (22), (23) or (40), and the current state of the crack length vector can be determined using (26) or (44).

A SPECIAL CASE

The fracture models for matrix crack in unidirectional fiber composites assume that the matrix crack is subject to an applied opening stress σ_a and a closure bridging stress σ_B which is related to the crack opening displacement (and its rate of change under fiber creep conditions) (Cox and Marshall, 1994; Marshall et al., 1985; Nair et al., 1991; Begley et al., 1995; Budiansky at al., 1986; Budiansky at al., 1995). An idealized concentric cylindrical cell have been commonly used to deduce the relationship between the bridging stress and the crack opening displacement under different fiber-matrix interface bonding conditions. While these authors have assumed that the bridging stress is defined by the relationship $\sigma_B = f\sigma_f$, El-Azab (1994) has defined the bridging stress based on some thermodynamical arguments, however, used the same micro mechanical model to implement that definition. For the sake of the present analysis, it is important to mention here that the unit cell micro mechanical models have assumed that debonding along each fiber-matrix interface is driven only by the local fiber stress, which is synonymous to saying that the debonding within a unit cell is described the cell own potential energy, and proceeds independent from debonding in other cell. To carry this argument further, let us be specific to the work of El-Azab (1994), since his formulation includes all energy terms used here in constructing the functionals P and $\Re$, although the argument applies to all the previous work. First, the assumption that debonding in different cells propagates independently under the influence of local fiber stress implies that the derivatives $\frac{\partial^2 P}{\partial \ell_i \partial \ell_j}$ or $\frac{\partial^2 \Re}{\partial \ell_i \partial \ell_j}$ are non-zero only for $i = j$, which means that the coefficient matrix in, say, Eq. (38) is diagonal, and the system speeds are basically uncoupled. Moreover, as far as the evolution of the matrix crack is concerned, the forms of the functionals P and $\Re$ remain basically the same. However, with regard to debonding cracks, since debonding is driven by the cell own fiber stress, these functionals are separately written for each cell, which is equivalent to assuming that the derivatives of the form $\frac{\partial^2 \delta_m}{\partial \ell_i \partial \ell_j}$ also drop, i.e., terms that include the applied stress ($\sigma_a \int_{-c}^{c} \delta_m \, dx$) are not included. Proceeding with these simplifications, Figures 3 and 4, respectively, show some representative results for the fiber stress and the evolution of the matrix and debonding cracks under fiber creep conditions and fixed applied load in unidirectional Nicalon-SiC composite. Debonding is shown only on one side of the matrix crack. No fiber failure is assumed and fibers are bridging the entire matrix crack. The loading conditions are: temperature = 1125 °C, initial crack length = 120 R, final crack length = 200 R, external stress = 1.75 $E_f \Delta\varepsilon_{th}$, E_f is the fiber modulus, $\Delta\varepsilon_{th}$ is the residual misfit strain, $E_f \Delta\varepsilon_{th}$=90MPa. The curves labeled 0 to 4 represent evolution prior to matrix crack propagation, while curves number 5 to 9 are evolution states at different times during matrix crack propagation. The interesting observation to be reported here is that, the continuous evolution of debonding around a matrix crack is a dictates the rate of dissipation by both interfacial friction and viscous deformation of the fibers. This ultimately impacts the time-to-propagation (incubation time) and the growth rates matrix cracks. A final remark to be made in this context is that, even under the restricting simplifications stated here it has been difficult to rigorously deal with this evolution problem. A method, whether analytical, numerical or hybrid, is yet to be developed to deal with this type of problem.

DISCUSSION AND CONCLUSIONS

The present work represents a first step in establishing a formal methodology to deal with debonding and matrix damage evolution in ceramic-matrix composites. This type of damage has a critical role in determining the behavior of matrix cracks and/or the ultimate path to failure in this class of materials. Eventhough the debonding contribution to the fracture energy of a ceramic-matrix fiber composites may be negligible, the nucleation and evolution of interface debonding cracks control the frictional energy loss, which is the main contribution to the toughness in such materials. Proper accounting for interface debonding processes is thus very important. Previous fracture mechanics or energy methods (Nair et al., 1991; Budiansky at al., 1986; Budiansky at al., 1995) have dealt mainly with debonding and matrix cracks separately, which is not proper. In composites with complex fiber architectures (e.g., in braided and woven composites), the nature of cracking is complex and matrix cracks are not straight. The associated debonding and slip damage can not be studied using simple micro mechanical models which have been developed for unidirectional composites. Moreover, matrix cracks themselves may be subject to strong frictional stresses due to different sorts of irregularities and depending on loading direction. This is where the method proposed here could be promising, even if it can only be implemented numerically.

Finite speeds for matrix and debonding cracks are obtainable as far as the solvability criterion for the system evolution equations (Eq. (7) or (8)) is met, that is; the second order partial derivatives (with respect to crack length) are strictly positive. It is to be mentioned here that, crack growth (under constant load) is unstable in the counter-example of a monolithic ceramic with no energy dissipative processes. It is also worth noting that the existence of a finite growth rate vector is a direct implication of two factors; direct bridging (terms containing fiber stress) and frictional dissipation (terms containing frictional stress). Stable evolution of the system means that finite increments in the externally imposed tractions (no creep case) or a decrement in fiber bridging forces and/or friction stress (creep case under constant external load) can only cause finite increments in the crack system lengths (areas). In general, all components of the vector $\vec{\ell}$ may not be growing during evolution. While this situation is not likely to be achieved in unidirectional composites with matrix cracks perpendicular to fibers, it is highly probable that this actually be the case with complex cracking configurations.

Although a unidirectional composite is assumed here to illustrate the method, it will only take a change of notation to generalize this method to treat most complex composite systems. Moreover, no specific assumptions have been stated as far as the nature of interfacial friction is considered, hence, constant or Coulomb friction models can be used with the present theory. In addition, no restricting assumptions about the material anisotropy, homogeneity, or any residual thermal or non-thermal stresses are included during the formulation, therefore, it could be applicable under different varieties of materials conditions. Fracture mechanics models, for example, assume that the material around the matrix crack will remain uniform and homogeneous during loading and evolution, even fibers debond, slip, creep or even fail, which may not be an accurate approximation. Energy methods ((Nair et al., 1991; Budiansky at al., 1986; Budiansky at al., 1995)) adopted what is called steady-state matrix cracking, which essentially proper for large matrix cracks and fibers which do not fail in the far crack wake. Other energy-based treatments (Aveston et al., 1971) did not consider the matrix crack length to be a parameter of the problem, which may not apply to real test specimens or structural components with cracks of finite length.

While the direction of interface debond cracks is pre-determined in composites with weak interfaces, matrix cracks may not propagate in a well behaved manner, i.e., could exhibit strong or weak kinking character. The present method can actually be implemented in such a way that crack kinking criteria are included and, in addition to evolution speeds, propagation paths can be determined. The method developed here is also amenable to consideration of finite fiber stress in the bridging zone, which means that growth instabilities associated with failure of fibers with finite strength or fiber failure due to excessive creep can also be analyzed. Other extrinsic factors such as residual stresses, notch size (unbridged zone size), overall specimen dimensions, etc., can also be included as bifurcation parameters which affect the growth stability, i.e., bifurcation parameter vector $\vec{\omega}$ must be included in the arguments of the functionals $P = P(\vec{\ell}, \vec{\lambda}; \vec{\omega})$ and $\Re = \Re(\vec{\ell}, \vec{\lambda}; \vec{\omega})$. An additional benefit which can be immediately recognized for the procedure outlined here is that, in a system with many cracks information about the effects of evolving sub-groups of cracks on the growth saturation of other sub-groups can be determined at different loading or creep stages. Also, this method can be used, along with an appropriate averaging methodology, to predict the overall stress-deformation response of composites with evolving damage under increasing loads. The formulation presented here is exact. Different approximations may then be utilized to model the composite and to obtain the results for the strain energy release rates and the field profiles.

ACKNOWLEDGMENT

This material is based upon work supported by the U. S. Department of Energy under grant number DE-FG03-91ER45115.

REFERENCES

Aveston, J., Cooper, G. A. and Kelly, A., 1971, " Single and Multiple Fracture," *Proceedings, The Properties of Fiber Composites*, National Physical Laboratory, Guildford, IPC Science and Technology Press Ltd., pp . 15-26.

Begley, M. R., Cox, B. N. and McMeeking, 1995, " Time-Dependent Crack Growth in Ceramic Matrix Composites With Creeping Fibers," *Acta Metallurgica et Materiialia*, in press.

Budiansky, Bernard, Hutchinson, John W. and Evans, Anthony, G., 1986, " Matrix Fracture in Fiber-Reinforced Ceramics," Journal of the Mechanics and Physics of Solids, Vol. 34, No. 2, pp. 167-189.

Budiansky, B., Evans, A. G. and Hutchinson, J. W., 1995, " Fiber-Matrix Debonding Effects on Cracking in Aligned Fiber Ceramic Composites," *International Journal of Solids and Structures*, Vol 32, No. 3/4, pp. 315-328.

Cox, B. N. and Marshall, D. B., 1994, " Concepts for Bridged Cracks in Fracture and Fatigue," *Acta Metallurgica et Materialia*, Vol. 42 No. 2, pp. 341-363.

Cox, B. N. and Marshall, D. B., 1987, " Stable and Unstable Solutions for Bridged Cracks in Various Specimens," *Acta Metallurgica et Materialia*, Vol. 39 No. 4, pp. 579-589.

Daniel, I. M. and Anastasspoulos, G., 1995, " Failure Mechanisms and Damage Evolution in Crossply Ceramic-Matrix Composites," International Journal of Solids and Structures, Vol. 32 No. 3/4, pp. 341-355.

El-Azab, Anter A. 1994 " Time-Dependent High-Temperature Fracture of Ceramic-Matrix Composites," Ph.D. Dissertation, University of California, Los Angeles, CA 90025, USA.

Golden, J. M. and Graham, G. A. C., 1990, "Energy Balance Criteria for Viscoelastic Fracture," *Quarterly of Applied Mathematics*, Vol 48, pp. 401-413.

Jakus, Karl and Nair, S. V., 1988, " Elevated Temperature Crack Growth in SiC Whisker-Reinforced Alumina," *Ceramic Engineering and Science Proceedings*, Vol. 9 No. 7-8, pp. 767-776.

McCartney, L. Neil, 1992a, " Mechanics for the Growth of Bridged Cracks in Composite Materials: Part I. Basic Principles," *Journal of Composites Technology & Research*, Vol. 14 No. 3, pp. 133-146.

McCartney, L. Neil, 1992b, " Mechanics for the Growth of Bridged Cracks in Composite Materials: Part II. Applications," *Journal of Composites Technology & Research*, Vol. 14 No. 3, pp. 133-146.

Marshall, D. B., Cox, B. N. and Evans, A. G., 1985, " The Mechanics of Matrix Cracking in Brittle-Matrix Fiber Composites," *Acta Metallurgica et Materialia*, Vol. 33 No. 11, pp. 2013-2021.

Marshall, D. B. and Cox, B. N., 1987, " Tensile Fracture of Brittle Matrix Composites: Influence of Fiber Strength," *Acta Metallurgica et Materialia*, Vol. 35 No. 11, pp. 2607-2619.

Maugin, Gerard A., 1992, " The Thermomechanics of Plasticity and Fracture," *Cambridge University Press*.

Nair, S. V. and Jakus, K., 1988, " Role of Glassy Interfaces in High Temperature Crack Growth in SiC Fiber Reinforced Alumina," *Ceramic Engineering and Science Proceedings*, Vol. 9 No. 7-8, pp. 681-686.

Nair, S. V. and Jakus, K. and Lardner, T. J., 1991, " The Mechanics of Matrix Cracking in Fiber Reinforced Ceramic Composites Containing a Viscous Interface," *Mechanics of Materials*, Vol. 12, pp. 229-244.

Nair, Shantikumar V. and Gwo, Tsung-Ju, 1993, " Role of Crack Wake Toughening on Elevated Temperature Crack Growth in a Fiber Reinforced Ceramic Composite," *Journal of Engineering Materials and Technology*, Vol 115, pp. 273-280.

Nair, Shanti V., 1990, " Crack-Wake Debonding and Toughness in Fiber- or Whisker-Reinforced Brittle-Matrix Composites," *Journal of the American Ceramic Society*, Vol 73, No. 10, pp. 2839-2847.

FIGURES

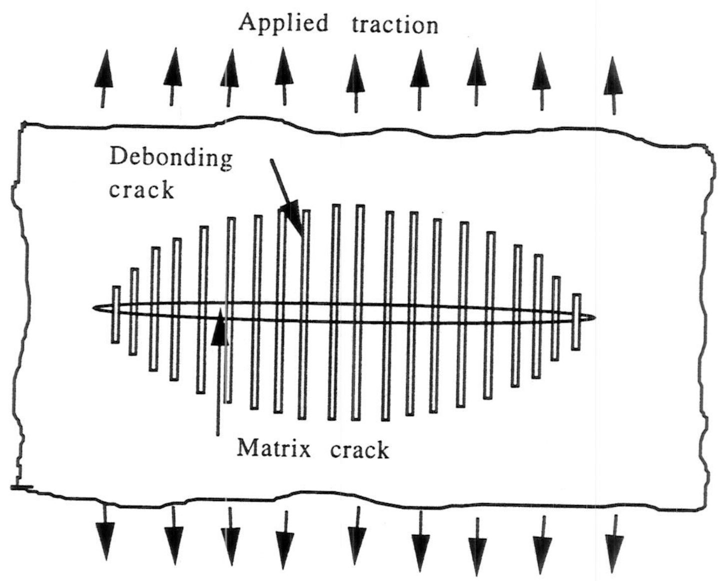

FIGURE 1: A MATRIX AND DEBONDING CRACK SYSTEM IN UNIDIRECTIONAL COMPOSITES.

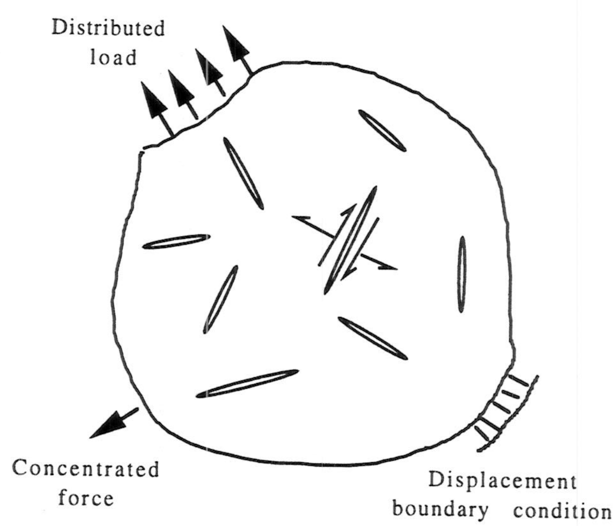

FIGURE 2: SCHEMATIC SHOWING MULTIPLY-CRACKED MATERIAL.

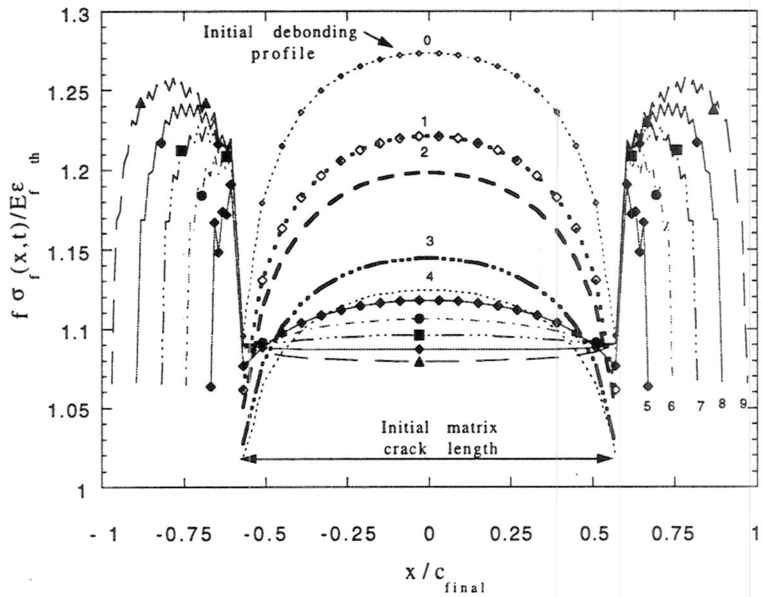

FIGURE 3: FIBER STRESS PROFILE DURING INCUBATION AND PROPAGATION OF A MATRIX CRACK IN NICALON-SIC COMPOSITE.

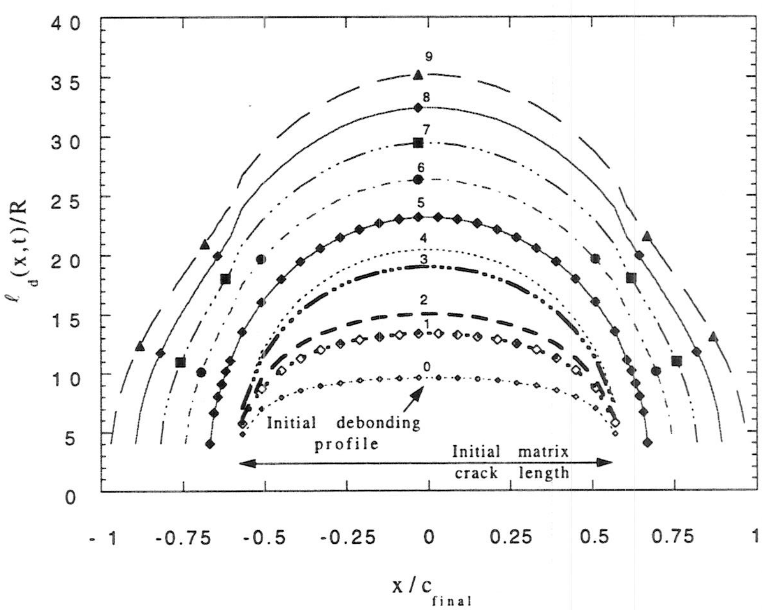

FIGURE 4: DEBONDING PROFILE DURING INCUBATION AND PROPAGATION OF A MATRIX CRACK IN NICALON-SIC COMPOSITE.